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INTEGRO-DIFFERENTIAL POTENTIALS FOR THE ANALYSIS OF A FRACTAL COVER PROPERTIES

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ABSTRACT

We have considered stationary electric and magnetic fields of fractal objects with the help of differintegration methods. The equations of fractal electrostatic and magnetostatic potential at integro-differential form are proved and introduced. The offered ideas can be useful for modification numeric fields modelling techniques to solve electromagnetic problems for real objects with material structure irregularities consideration.

INTRODUCTION

There are a number of modern numeric electromagnetic fields modelling techniques. Finite-Difference Time-Domain (FDTD), Method of Moments (MoM), and Finite Element Method (FEM) are the most popular and useful methods. The general essence of these methods consists in division of the approached ideal model of real physical object into elementary components the fields for which can be found on the basis of the classic Maxwell's theory. But the classic electromagnetic theory in which the object's geometry definition is based on concepts of point, line and plane (ideal objects) become impenetrable when the explanation of the field distribution of roughness fractal surfaces is needed. The examples of differintegration methods application for such complex electromagnetic problems were originally shown at [1].

MAIN PART

Based on idea, that the fractal sets are adequate geometrical model for irregular contours and surfaces. Let us consider prefractal covering compact set, which is our proposed model of a contour with a current, as a limit of monotonously growing sequence of covering compact sets with the corresponding sequence of rising contours $l \subset l_1 \subset \ldots \subset l_i \subset \ldots\ l_n$. The current near by l_n contour can be found as:

$$I(x) = \frac{1}{A(l,n)} \int_a^x dl_{n-1} \int_a^{l_{n-1}} dl_{n-2} \ldots \int_a^{l_1} \rho(x')dx' .$$

The obtained repeated integral may be summarize by Riemann's-Liouville's definition of fractional integral

$$({}_aD_x^{\alpha}\rho)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\rho(x')}{(x-x')^{1-\alpha}} dx' . \qquad (1)$$

As a result of usage fractal contour (surface) covering by compact sets, we have reduced a problem to construction a smoothing Hausdorf's measure on a physically prefractal layer with differintegration of an uniform electric charge density on a fractal point set projection to a smooth-faced segment [2].

Using of α-order differintegral definition of fractal electric charge density $\rho^{\alpha}(\mathbf{r}')$ located at $\mathbf{r}'$ Eq. (1) enables to formulate the equations of classical electrostatics in terms of α-characteristics. In analogy with the fractal electrostatic case, we may define fractal electric current density $\mathbf{j}^{\alpha}(\mathbf{r})$ for magnetostatic field analysis. So the laws of electrostatics and magnetostatics for fractal objects can be summarized in two pairs of time-independent, uncoupled vector differintegral equations, namely the equations of fractal electrostatics

$$\nabla\cdot\mathbf{E}^{\alpha}(\mathbf{r})=\frac{\rho^{\alpha}(\mathbf{r})}{\varepsilon_0}, \tag{2}$$

$$\nabla\cdot\mathbf{E}^{\alpha}(\mathbf{r})=0,$$

and the equations of fractal magnetostatics

$$\nabla\times\mathbf{H}^{\alpha}(\mathbf{r})=\mu_0\mathbf{j}^{\alpha}(\mathbf{r}), \tag{3}$$

$$\nabla\times\mathbf{H}^{\alpha}(\mathbf{r})=0.$$

The electrostatic field $\mathbf{E}^{\alpha}(\mathbf{r})$ is irrotational and it may be expressed in terms of the gradient of a scalar field. If we denote this scalar field by $-\phi^{\alpha}(\mathbf{r})$, we get $\mathbf{E}^{\alpha}(\mathbf{r})=-\nabla\phi^{\alpha}(\mathbf{r})$. Taking the divergence of this and using Eq. (2), we obtain Poisson's equation

$$\nabla^2\phi^{\alpha}(\mathbf{r})=-\nabla\cdot\mathbf{E}^{\alpha}(\mathbf{r})=-\frac{\rho^{\alpha}(\mathbf{r})}{\varepsilon_0}. \tag{4}$$

The solution of Eq. (4)

$$\phi^{\alpha}(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{\rho^{\alpha}(\mathbf{r})}{|\mathbf{r}-\mathbf{r}'|}dv'+c, \tag{5}$$

where the integration is taken over all source points $\mathbf{r}'$ at which the charge density $\rho^{\alpha}(\mathbf{r}')$ is non-zero and c is an arbitrary quantity which has a vanishing gradient. The scalar function $\phi^{\alpha}(\mathbf{r})$ in Eq. (5) above is called the fractal electrostatic scalar potential.

Consider the equations of magnetostatics Eq. (3) we got fractal magnetostatic vector potential with definition from $\mathbf{B}^{\alpha}(\mathbf{r})=\nabla\times\mathbf{A}^{\alpha}(\mathbf{r})$ as

$$\mathbf{A}^{\alpha}(\mathbf{r})=\frac{\mu_0}{4\pi}\int_{V'}\frac{\mathbf{j}^{\alpha}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dv'+\mathbf{a}(\mathbf{r}), \tag{6}$$

where $\mathbf{a}(\mathbf{r})$ is an arbitrary vector field whose curl vanishes.

CALCULUS EXAMPLES

As an example of using obtained expressions (Eq. 4-6) at FEM methods applications let us consider following problem

$$\nabla^2\phi=-1 \text{ inside } B,\ D^{\alpha}\phi\big|_A=C,\ \phi\big|_B=0, \tag{7}$$

where A border is a circle defined as $\{x \cdot \cos(t);\ y \cdot \sin(t);\ 0 \le t \le 2\pi\}$ and B circle is infinity approximation defined as $\{x \cdot 5\cos(t);\ y \cdot 5\sin(t);\ 0 \le t \le 2\pi\}$.

Fractal boundary condition $D^{\alpha}\phi\big|_A = C$ translates into $\phi^{(\alpha)}\big|_A = C/(x-b)^{1-\alpha}$, $1 \le \alpha \le 2$.

The problem (7) was solved for $b=-5$, $C=1$ parameters by finite element method with the help of FreeFEM+ software. The obtained potential surface for $\alpha = 1.7$ is shown in Fig. 1. We demonstrate the XOZ-plane cut of potential distribution for α-parameter varied from 1 up to 2 in Fig. 2.

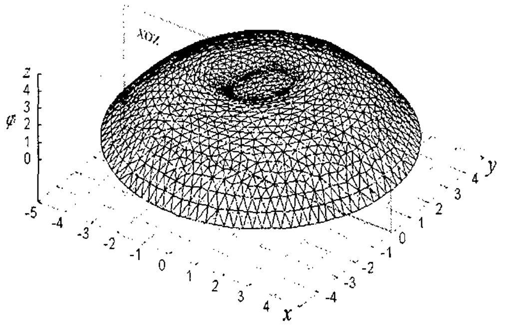

Fig. 1. 3D potential distribution for $\alpha = 1.7$

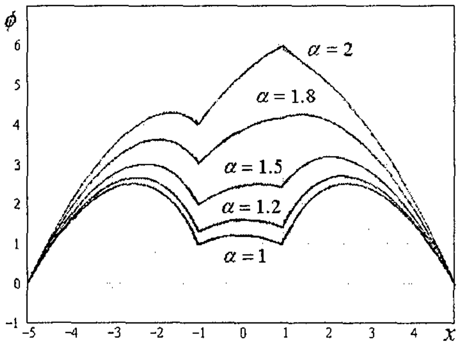

Fig. 2. Potential distribution for various α-parameter

The problem (7) with $\alpha = 1$ and $\alpha = 2$ parameters of fractal boundary condition is equivalent of Dirichlet.

More interesting that at $\alpha = 1.5$ calculation result is equal to solving of mixed problem with Dirichlet and Neumann boundary conditions like $(\partial\phi/\partial\mathbf{n})\big|_A + \phi\big|_A = x$.

CONCLUSIONS

The application of fractional calculus enables to formulate the electromagnetic potential equations in terms of α-characteristics. Formally proposed equations agree with the classical, at imposing on α-characteristics additional boundary conditions.

Numeric decision of typical problem of potential theory has shown that definition of boundary conditions in the integro-differetial form generalizes classical boundary conditions and is adequate for fractal surfaces with various levels of roughness. The offered ideas assumes to remove difficulties in investigations of singular distributions and can be useful for modification numeric fields modeling techniques to solve electromagnetic problems for real objects with structure irregularities consideration.

REFERENCES

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